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NONCOMPACT RIEMANNIAN MANIFOLDS WITH DISCRETE SPECTRA

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Introduction

This paper grew out of an attempt to characterize those complete noncompact surfaces of revolution in $\mathbf{R}^3$ whose Laplacians have a discrete spectrum. Since a surface of revolution is a cylinder of the form $\mathbf{R}^1 \times S^1$, it is only natural to try the separation of variables method to study this problem. Now the spectrum of S^1 is discrete, and therefore the space L^2 of the original surface splits into a direct sum of infinitely many copies of $L^2_\mu(\mathbf{R}^1)$, one for each eigenvalue of the Laplacian on S^1, repeated according to multiplicity. Here μ is a measure which varies with the surface under consideration; likewise the original Laplacian decomposes into a direct sum of 'component' operators A_n on $\mathbf{R}^1$, symmetric relative to μ (in fact essentially selfadjoint on $C_0^\infty(\mathbf{R}^1)$), so that spectral questions concerning the Laplacian become questions about each one of these components. It turns out, as it often happens, that the study of the combined effect of these operators is in no way facilitated by the fact that one is only considering surfaces of revolution in $\mathbf{R}^3$, say. That is why we decided to abstract and present here the common features which always occur when studying a manifold Z which, at least outside some compact subset, can be expressed as a product $X \times Y$, and whose Riemannian structure is reasonably well adapted to this decomposition. For the exact conditions we require, see the beginnings of §§3 and 4. If the fibre Y is discrete, i.e., if its Laplacian has a discrete spectrum, one is led to operators A_n on X, just as in the case of a cylinder. These operators have a simple description in terms of the geometric data pertaining to the manifold under consideration; in other words, although the A_n occur as a result of 'separating variables', they are intrinsic objects, independent of coordinate choices. Our main result (Theorem 3.3), is that all these operators, but the first one A_1, can be disregarded as far as the discreteness of Z is concerned. What we have here is truly a reduction theorem, the end product being the manifold X together with the operator A_1 on it, which for expediency we refer

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to as an A_1-manifold. Theorem 3.3 states that the discreteness of the (Riemannian) manifold Z is equivalent to the discreteness of the A_1-manifold X. When Z is a surface of revolution, X is 1-dimensional and A_1 is an operator which can be analyzed completely, leading to the sought characterization (Theorem 4.1). In particular we find that a manifold of this type can be discrete only if its total volume (i.e., its surface area) is finite, although this condition is by no means sufficient. For example any surface generated by the rotation around the x-axis in $\mathbf{R}^3$ of $\gamma(x) = |x|^{-\alpha}$, $|x|$ large, has a finite volume if $\alpha > 1$; none of these surfaces is discrete though. Among the surfaces generated by decaying exponentials of the form $\gamma(x) = e^{-|x|^\alpha}$, $|x|$ large, the only ones which are discrete are those for which $\alpha > 1$. Finally, we use the reduction method outlined above to show that surfaces of revolution are rather exceptional in so far as they must have a finite volume in order to be discrete. For each dimension n greater than 2 we construct (Proposition 4.3) a whole class of n-dimensional complete discrete manifolds whose volumes are infinite.

The organization of the paper is as follows: in §1 we study an abstract version of an operator A which is the direct sum of a sequence of operators A_n and obtain a discreteness criterion (Corollary 1.4) which is all we need from this section later on. A complete description of the relationship between the spectrum of A and the spectra of the A_n is given in Theorem 1.3. §2 is a miscellany of assorted statements and results concerning second order elliptic differential operators on a noncompact manifold most of which have appeared in one form or another in the literature. We have included it because we know of no source which presents these theorems in the generality needed here and/or in the spirit of this article. §§3 and 4 constitute the heart of the paper, but they have already been essentialy discussed.

1. Infinite direct sums of operators

Let H be a Hilbert space. If A is a densely defined symmetric operator on H with domain D, we shall denote by D^1 the domain of the adjoint of A. This adjoint is an extension of $A \restriction D$, which for simplicity will also be called A. D^1 is a Hilbert space relative to the graph norm of A; the closure of D in this space is the domain of the minimal extension of A and will be denoted by D^0. Finally if $A \restriction D$ is semibounded, then D^F will stand for the domain of the Friedrichs extension of A. Now suppose H is the complete direct sum $\hat{\bigoplus} H_n$ of a sequence of mutually orthogonal subsapces H_n, and let P_n be the corresponding projections. In addition suppose that on each H_n there is given a densely defined symmetric operator with domain D_n. Let $A = \bigoplus A_n$ be the (algebraic) sum of the $A_n \restriction D_n$ with domain $\oplus D_n$. Our aim in this section is to relate properties of the A_n to corresponding properties of A. To begin with, we remark that A is nonnegative if and only if all the A_n are.

Proposition 1.1. *The space D^j ($j = 0, 1$ or F in case all the A_n are nonnegative) is the complete direct sum $\hat{\oplus} D_n^j$. The corresponding orthogonal projections are just the restrictions of the P_n's to D^j; moreover all the diagrams of Hilbert spaces and continuous maps*

$$\begin{array}{ccc} D^j & \xrightarrow{A} & H \\ P_n \downarrow & & \downarrow P_n \\ D_n^j & \xrightarrow[A_n]{} & H_n \end{array}$$

commute.

The proof of all this is standard and will be left to the reader.

A consequence of Proposition 1.1 is that the defect index $d_z(A)$ of $A \restriction D$ at $z \in \mathbf{C}$ can be calculated in terms of the corresponding defect indices of the A_n. Here $d_z(A)$ stands for the dimension of $\operatorname{Ker}(A - zI)$ on D^1.

Corollary 1.2. $d_z(A \restriction D) = \Sigma_n d_z(A_n \restriction D_n)$. *In particular A is essentially selfadjoint on D if and only if all the $A_n \restriction D_n$ are.*

Proof. Indeed, Proposition 1.1 implies that

$$\operatorname{Ker}(A - zI \restriction D^1) = \overline{\bigoplus_n \operatorname{Ker}(A_n - zI \restriction D_n^1)},$$

the closure being taken in D^1.

We shall now turn to a description of the spectrum $S(A)$ of A in terms of the spectra of the A_n. We recall [5, p. 7] that a complex number λ is in the continuous spectrum $C(A)$ of A iff there exists a characteristic sequence for $A - \lambda$, i.e., a bounded sequence u_m in the domain of A having no convergent subsequence such that $Au_m - \lambda u_m \to 0$. For simplicity we shall restrict ourselves to the selfadjoint case which is all we need in our applications. More precisely, we shall assume that the $A_n \restriction D_n$ are either all essentially selfadjoint ($D_n^0 = D_n^1$), or all nonnegative. In the first case $A \restriction D$ is essentially selfadjoint; its minimal closed extension is selfadjoint. In the latter the Friedrichs extensions of A, A_n are selfadjoint. In all cases we shall only consider the selfadjoint extensions just mentioned. The simplification alluded to above results from the fact that the residual spectrum of an arbitrary selfadjoint extension operator T is empty, and consequently its spectrum $S(T)$ is the set theoretical union of the continuous spectrum $C(T)$ and the point spectrum $P(T)$. Since an eigenvalue of infinite multiplicity is necessarily in the continuous spectrum, it follows that in the absence of the latter, $S(T)$ consists of a sequence of isolated eigenvalues of finite multiplicity with no accumulation point. Conversely if this is the case, then the continuous spectrum is lacking. Following common (if somewhat confusing) usage, we shall refer to this by saying that the spectrum of T is *discrete*.

Our next result establishes the sought relations between the spectra of the A_n and the spectrum of A. To this effect we introduce the limit set L_∞ of the family $P(A_n)$ of point spectra of the A_n, as the set of limits of all possible convergent sequences $\lambda_{n_i} \in P(A_{n_i})$ where n_i is a strictly increasing sequence of positive integers. Equivalently, $\lambda \in L_\infty$ iff for every neighborhood N of λ there are infinitely many n's such that $N \cap P(A_n) \neq \varnothing$.

Theorem 1.3.

$$\text{(i)} \quad P(A) = \bigcup_n P(A_n).$$

$$\text{(ii)} \quad C(A) = \overline{\bigcup_n C(A_n) \cup L_\infty}.$$

$$\text{(iii)} \quad S(A) = \overline{\bigcup_n S(A_n)}.$$

(It is understood that the domains of A, A_n are either $D^0 = D^1$, $D_n^0 = D_n^1$ or D^F, D_n^F respectively, as explained before.)

Proof. Part (i) as well as the inclusion $C(A_n) \subseteq C(A)$ are direct consequences of Proposition 1.1. That $L_\infty \subseteq C(A)$ follows from observing that a sequence u_{n_i} of unit vectors satisfying $A_{n_i}u_{n_i} = \lambda_{n_i}u_{n_i}$ for some sequence λ_{n_i} which converges to a number λ, is characteristic for $A - \lambda I$; observe that the u_{n_i} are mutually orthogonal if the sequence n_i is strictly increasing. Since the continuous spectrum of a closed operator is closed [5, p. 8], it follows from the above that $\overline{L_\infty \cup \bigcup_n C(A_n)} \subseteq C(A)$. Before proving the other half of (ii) we go to (iii). We have already established that $S(A_n) = C(A_n) \cup P(A_n) \subseteq S(A)$. Since the spectrum of an operator is closed, it is enough to show that if $\lambda \notin \overline{\bigcup_n S(A_n)}$, then $\lambda \notin S(A)$. Let $\varepsilon > 0$ be smaller than the distance from λ to each one of the spectra $S(A_n)$. If $R(A_n; \lambda)$ is the resolvent of A_n at λ, then it is well known (see e.g. [8, p. 272]) that $R(A_n; \lambda) < 1/\text{dist}(\lambda, S(A_n)) < 1/\varepsilon$. Let u be in the domain of A. By Proposition 1.1, $u = \Sigma_n u_n$, where u_n is in the domain of A_n, the series converging in the Hilbert space D^1. Therefore $Au = \Sigma_n Au_n = \Sigma_n A_n u_n$ (in H), and

$$\|Au - \lambda u\|^2 = \sum_n \|A_n u_n - u_n\|^2 \geqslant \varepsilon^2 \sum_n \|u_n\|^2 = \varepsilon^2 \|u\|^2.$$

Since $\lambda \notin P(A)$ it follows that λ is in the resolvent set of A which proves (iii). We finally turn to (ii). Suppose there exists a λ in $C(A)$ which is not in $\overline{L_\infty \cup \bigcup_n C(A_n)}$. It follows that some neighborhood N of λ intersects at most finitely many $P(A_n)$, $n = 1, 2, \cdots, k$, say. We may further assume that

$N \cap C(A_n) = \varnothing$ for all n. Now consider the splitting $A = L \oplus M$ where $L = \oplus_1^k A_N$, and M is the closure of the algebraic sum $\oplus_{k+1}^{\infty} A_n$, the domains of L, M being the obvious ones in each of the two cases under consideration. It is clear that $C(A) = C(L) \cup C(M)$. Since λ is not in $\oplus_1^k C(A_n) = C(L)$ it follows that $\lambda \in C(M) \subseteq S(M)$. Now part (iii) already proved, applied to M, implies that

$$\lambda \in \bigcup_{k+1}^{\infty} P(A_n) \cup \overline{\bigcup_{k+1}^{\infty} C(A_n)} = S(M),$$

which is a contradiction.

As a consequence of Theorem 1.3 we have

Corollary 1.4. *Suppose A, A_n are nonnegative operators. Let $\lambda_1(A_n)$ be the least point of the spectrum of the Friedrichs extension of A_n. Then the following two conditions are, together, necessary and sufficient for the Friedrichs extension of A to have a discrete spectrum:*

(i) *The spectra of the A_n are all discrete;*

(ii) *$\lambda_1(A) \to \infty$ as $n \to \infty$.*

2. *A*-manifolds

Although our main interest in this paper are Riemannian manifolds we shall see in §3 that we need to consider a slightly more general structure which we now pass to explain. Let X be a smooth (C^∞) manifold which for simplicity we shall assume to be connected. A real second order elliptic differential operator with smooth coefficients and a positive definite (principal) symbol will be referred to as an *operator*, for short. Here the sign convention for the symbol is made consistent with the requirement that at (x, ξ) the symbol of $D_j = i^{-1}\partial/\partial x_j$ in $\mathbf{R}^n$ be ξ_j (see e.g., [7, p. 30] for an intrinsic definition). Thus the Laplacian $-\Delta = -\sum \partial^2/\partial x_j^2$ in $\mathbf{R}^n$ is such an operator with symbol $|\xi|^2$. More generally any Riemannian structure on X determines an operator, namely its associated Laplacian, which in turn uniquely determines the given structure (on T^*X) via its symbol. Now let A be an arbitrary operator with symbol $a(x, \xi)$ at $(x, \xi) \in T_x^*(X)$. We shall refer to the pair (X, A) as an A-manifold. By definition of operator, $a(x, \xi)$ is the quadratic form of a Riemannian metric on $T^*(X)$ which in local coordinates $x = (x_2, x_2, \cdots, x_n)$ has the form $a(x, \xi) = \sum a^{ij}(x)\xi_i\xi_j$. Let $-\Delta$ be the corresponding Laplacian. Since A and $-\Delta$ have the same symbol, $A + \Delta$ is a first order differential operator $U + c$ with U a real vector field and $c = A(1)$ a real valued function both uniquely determined by A in such a way that one has

(1) $$A = -\Delta + U + c.$$

In other words, an A-structure on X is equivalent to a triple of the form (Riemannian metric; real vector field; real valued function) and we shall identify one with the other.

Next, we study the question of symmetry of A relative to the inner product on $C_0^\infty(X)$ given by some (smooth, positive) density μ on X. Some remarks concerning densities of this type are in order here. Let us recall that a density is a measure which on a coordinate patch can be represented in the form $\mu(x)\,dx$, with $\mu(x)$ strictly positive and smooth. Alternatively one can think of μ as a section of the line bundle whose transition functions are the absolute values of the Jacobians of the coordinate changes. Thus it is clear that if ν is another section of this bundle the quotient ν/μ is a well-defined function on X. Also, since the bundle of densities is trivial, any density is of the form $f\mu_0$ with μ_0 a fixed given one. Now let V be a vector field on X. We recall that it is usual to denote by $\operatorname{div}_\mu V$ the unique function which for all μ in $C_0^\infty(X)$ satisfies

$$\int_X \varphi \operatorname{div}_\mu V \cdot \mu(dx) = -\int_X V(\varphi)\mu(dx).$$

In local coordinates, if $V = \sum V^i(x)\partial/\partial x_i$ and $\mu = \mu(x)\,dx$, then $\operatorname{div}_\mu V = \sum \mu^{-1}(x)(\partial/\partial x_i)(\mu(x)V^i)$.

We are ready to go back to the question of symmetry. Consider a triple of the form $(a(x, \xi); \mu; c)$ where now μ is a density, and $a(x, \xi)$, c are as above. If grad stands for the gradient operation relative to the metric $a(x, \xi)$, then the expression

(2) $$A = -\operatorname{div}_\mu \operatorname{grad} + c$$

defines an operator which is in fact characterized as the unique operator with symbol $a(x, \xi)$, symmetric relative to μ and such that $A(1) = c$. Writing A in the form (1), it is easy to see that the vector field occurring there satisfies

(3) $$U = \operatorname{grad} \log(\mu/\mu_0),$$

where μ_0 is the Riemannian volume element associated with $a(x, \xi)$. Conversely, if $A \leftrightarrow (a(x, \xi); U; c)$ is an operator symmetric relative to some density μ, then this density satisfies (3), and it is therefore determined by A up to a positive multiplicative constant. Thus we have a mapping

$$(a(x; \xi); \mu; c) \to (a(x, \xi); U; c) \leftrightarrow A$$

which is one-one except for the action of $\mathbf{R}_+$ on μ, and whose image is the set of operators which are symmetric relative to some density on X. There is then little ambiguity in referring to such an operator simply as a *symmetric operator* and we shall do so. It is interesting to note that for a general operator to be symmetric, there are local as well as global obstructions, namely, that the 1-form determined by U be closed and have vanishing periods. For example on the circle every operator can be written in the form $A = -\mathrm{a}d^2/d\theta^2 + \mathrm{b}d/d\theta + c$ with $a > 0$, b, c, real and periodic. The 1-form alluded to above is $ba^{-1}d\theta + \frac{1}{2}d\log(a)$, and consequently the necessary and sufficient condition for A to be symmetric is the vanishing of the integral $\int_0^{2\pi} ba^{-1}d\theta$. In what follows we shall confine ourselves to the symmetric case. We just mention that the nonsymmetric case has been the object of recent papers by Donsker and Varadhan [1], [2] where a variational formula for the "vertex" of the spectrum is given.

Let X be an A-manifold with A of the form (2). We shall consider $A \restriction C_0^\infty(X)$ as an unbounded operator on $L^2(X; \mu)$. A is semibounded if and only if the Rayleigh "quotient"

$$\lambda_1 = \lambda_1(A) = \inf_{C_0^\infty(X)} (A\varphi, \varphi)/(\varphi, \varphi)$$

is finite. In this case λ_1 is the infimum of the spectrum of the Friedrichs extension of A. We note in passing that the *value* of A, $(A\varphi, \varphi)/(\varphi, \varphi)$ is independent of the measure chosen relative to which A is symmetric. Also, the set of these values for φ with support in a fixed open nonempty subset of X contains with every λ the whole segment $[\lambda, \infty)$; therefore the statement 'A is semibounded' can only mean 'semibounded below'.

Theorem 2.1. *Suppose there exist a strictly positive function $f \in C^\infty(X)$ and a number $\lambda \in \mathbf{R}$ such that*

$$f^{-1}Af(x) \geqslant \lambda, \qquad x \in X. \tag{3}$$

Then A is semibounded and in fact $\lambda_1 \geqslant \lambda$. Conversely if $\lambda_1 > \lambda$, then there exists some positive $f \in C^\infty(X)$ which satisfies (3). *As a result, λ_1 has the following alternative characterization*:

$$\lambda_1 = \sup_f \inf_{x \in X} (f^{-1}Af)(x), \tag{4}$$

the supremum being taken over all strictly positive functions $f \in C^\infty(X)$.

Proof. This result appears to be a folklore theorem. The characterization (4) is a particular case of the main theorem in [1], where X is assumed to be *compact*. Since we know of no reference which applies to our case we shall give a proof, although some details will be left out.

First of all we note that the inequality $\lambda_1 \geqslant \lambda$ is trivial if f in (3) can be taken to be the constant function 1. Indeed in this case A is the sum of the nonnegative operator $-\mathrm{div}_\mu$ grad and multiplication by a function, namely

$A(1)$, which is bounded from below by λ. The general case can be reduced to the one just treated by means of the following trick: multiplication by f is an isometry from $L^2(X, f^2\mu)$ onto $L^2(X, \mu)$ which preserves $C_0^\infty(X)$ and intertwines the operators A and $\tilde{A} = f^{-1} \circ A \circ f$. The reduction alluded to above is accomplished by noting that $\lambda_1(A) = \lambda_1(\tilde{A})$ and that $\tilde{A}(1)(x) = f^{-1}Af(x)$.

The converse is an existence theorem for the differential inequality (3). Now if $\lambda_1 > \lambda$, then in particular A is semibounded and λ_1 is the bottom of the spectrum of its Friedrichs extension. It is well known that if λ_1 is an eigenvalue, its multiplicity is 1 and the corresponding eigenfunction can be taken to be everywhere positive as well as smooth due to the ellipticity of A. Such an eigenfunction is certainly a solution to our inequality. Again, the general case can be reduced to the above by a well-known trick (see e.g., [3], [12]) which consists of subtracting from A a nonnegative $\varphi \in C_0^\infty(X)$ in such a way that

$$\lambda < \lambda_1(A - \varphi) < \lambda_1(A).$$

Since such a perturbation is compact relative to A, the continuous spectrum remains unchanged, and consequently $A - \varphi$ does have $\lambda_1(A - \varphi)$ in its point spectrum. The corresponding positive eigenfunction f satisfies

$$f^{-1}Af(x) \geqslant f^{-1}Af(x) - \varphi(x) = \lambda_1(A - \varphi) > \lambda.$$

This concludes the proof of Theorem 2.1.

We now turn to the question of discreteness of the spectrum of the Friedrichs extension of an operator A which is symmetric and semibounded. In what follows A is a fixed operator on X, but we shall also consider A as an operator on open subsets G of X. For the rest of this section we shall change somewhat our earlier notation: $\lambda_1(G)$ will stand for the Rayleigh quotient of A as an operator on $C_0^\infty(G)$. Likewise, we set $\kappa(G)$ for the infimum of the continuous spectrum of the Friedrichs extension of $A \restriction C_0^\infty(G)$ with the understanding that $\kappa(G) = +\infty$ if the continuous spectrum is empty.

Theorem 2.2. *The operator A has a discrete spectrum if and only if*

$$\lim_K \lambda_1(X - K) = \sup_K \lambda_1(X - K) = +\infty, \tag{5}$$

where K runs through the family of compact subsets of X directed by inclusion. More generally, $\kappa(X) = \lim_K \lambda_1(X - K)$.

Proof. Suppose there exists $\lambda_0 < +\infty$ such that for every compact subset K of X one has $\lambda_1(X - K) < \lambda_0$. We shall show that there is a point in the continuous spectrum not greater than λ_0. Indeed our assumption implies the existence of a sequence of functions $\varphi_j \in C_0^\infty(X)$ with mutually disjoint supports such that $(A\varphi_j, \varphi_j) < \lambda_0\|\varphi_j\|^2$. It is clear that this inequality persists on the linear space generated by the φ_j. According to [5, Theorem 13, p. 15], such an inequality on an infinite dimensional subspace implies that the part of the spectrum to the left of λ_0 is either an infinite set or contains at least one eigenvalue of infinite multiplicity. Our assertion follows from this and the fact that the spectrum of A is bounded below.

For the converse we need

Lemma 2.3. *If K is a compact subset of X, then $\kappa(X - K) \leqslant \kappa(X)$.*

Theorem 2.2 is an immediate consequence of the lemma, for one has

$$\lim_K \lambda_1(X - K) = \sup_K \lambda_1(X - K) \leqslant \sup_K \kappa(X - K) \leqslant \kappa(X).$$

Now Lemma 2.3 is a weak version of the so called "decomposition principle" that can be found in [5, p. 59], or in a different context in [11, p. 192]. Since our assumptions differ from these authors', we shall give a proof for the sake of completeness. Let λ be a point in the continuous spectrum of A as an operator on X. We will show that λ is in the continuous spectrum of A as an operator on $X - K$. If u_j is a characteristic sequence for $A - \lambda$, it is well known that u_j belongs to the Sobolev space $H_2^{\mathrm{loc}}(X)$; since this sequence is bounded in the graph norm of A, Rellich's theorem implies that on the compact set K, u_j may be taken to be convergent (in L^2), and in fact convergent to zero there, since as it is not difficult to see u_j could have been taken weakly convergent to zero in the first place; for example, orthogonal. By another modification of our choice we may even assume that u_j converges to zero in L^2 of a relatively compact neighborhood U of K. If $\alpha \in C_0^\infty(U)$ is identically one on a neighborhood of K and $\beta = 1 - \alpha$, we claim that βu_j is characteristic for $A - \lambda$ on $X - K$; we need to show that βu_j is in the domain of the Friedrichs extension of A on $X - K$ and that $(A - \lambda)\beta u_j \to 0$ in L^2. Now the Friedrichs extension has for domain the set $H(A) \cap D^1(A)$, where $H(A)$ is the completion of C_0^∞ in the norm $[\varphi]^2 = (A\varphi, \varphi) + \gamma\|\varphi\|^2$, γ sufficiently large, and $D^1(A) = \{u \in L^2 | Au \in L^2\}$, Au being understood in the distributional sense. To see that $\beta u_j \in H(A)$, it is enough to show that multiplication by β is continuous on C_0^∞ with the norm of $H(A)$. In fact if $\varphi \in C_0^\infty$, then

$$[\beta\varphi]^2 = (A(\beta\varphi), \beta\varphi) + \gamma\|\beta\varphi\|^2 = (|\beta|^2 A\varphi, \varphi) + (\varphi(A - A(1))\beta, \beta\varphi)$$
$$-2(\operatorname{grad}\beta \cdot \operatorname{grad}\varphi, \beta\varphi) + \gamma\|\beta\varphi\|^2.$$

The first two terms of the right-hand side are clearly dominated by $[\varphi]^2$, and the third one is dominated by $\|\operatorname{grad}\varphi\|^2 + \|\varphi\|^2$. Now

$$\|\operatorname{grad}\varphi\|^2 = \int \operatorname{grad}\varphi \cdot \operatorname{grad}\bar{\varphi}\mu(dx) = -\int \bar{\varphi}\operatorname{div}_\mu \operatorname{grad}\varphi\mu(dx)$$
$$= (A\varphi, \varphi) - (A(1)\varphi, \varphi).$$

This shows that multiplication by β is continuous in $H(A)$ provided that $A(1)$

is bounded below. If $A(1)$ is not bounded below we proceed as in the proof of Theorem 2.1, replacing A by $\tilde{A} = f^{-1} \circ A \circ f$ where f is any strictly positive function such that $f^{-1}(x)Af(x)$ is bounded below, which exists in view of the fact that A is bounded below. Finally, that $(A - \lambda)\beta u_j$ is in $L^2(X - K; \mu)$ and in fact tends to zero, follows from the identity

$$(A - \lambda)(\beta u_j) = \beta(A - \lambda)u_j - u_j \operatorname{div}_\mu \operatorname{grad} \beta - 2 \operatorname{grad} u_j \cdot \operatorname{grad} \beta.$$

Indeed it is well known from elliptic theory that $\operatorname{grad} u_j \in H_1^{\mathrm{loc}}$ and it is bounded there; by Rellich's theorem it is relatively compact in L^2 of any compact subset of X, and in fact tends to zero there because it does so in the sense of distributions. Since $\operatorname{grad} \beta$ has compact support and $(A - \lambda)u_j \to 0$, the lemma is proved and so is the theorem.

Combining Theorems 2.1 and 2.2 we obtain the following criterion for discreteness:

Corollary 2.4. *A is discrete if there exists a positive function f (defined in the complement of some compact subset) such that*

$$f^{-1}(x)Af(x) \to +\infty \quad \text{as } x \to \infty \tag{6}$$

in the sense of the Alexandroff compactification of X. Conversely if A is discrete and λ is an arbitrary positive constant, there exist a compact subset K of X and a positive function $f \in C^\infty(X - K)$ such that $f^{-1}(x)Af(x) > \lambda$ on $X - K$.

We remark that the criterion given by Corollary 2.4 is a generalization of the following well-known result (see e.g., [5, p. 146, Theorem 1]): $A = -\Delta + q$ on $\mathbf{R}^n$ has a discrete spectrum if $q(x) \to \infty$ as $x \to \infty$. In fact $q(x) = f^{-1}Af(x)$ with $f \equiv 1$. This result extends of course to the Laplacian on any Riemannian manifold.

Corollary 2.5 (*Decomposition principle*). *A is discrete if and only if A is discrete as an operator on the complement of any compact set K_0.*

Proof. The limit (5) is the same for X and $X - K_0$.

Corollary 2.6. *If A is discrete and q is a nonnegative function, then $A + q$ is also discrete.*

Proof. Obvious, since the Rayleigh quotient of $A + q$ on the complement $X - K$ of any compact set K is not smaller than $\lambda_1(X - K)$.

3. A reduction principle

We shall consider Riemannian manifolds Z having the following structure:

Z can be expressed as a product $X \times Y$, where X, the base, and Y, the fibre, are Riemannian manifolds, with Y discrete, such that at each point $z = (x, y)$, the submanifolds $X_y = X \times \{y\}$ and $Y_x = \{x\} \times Y$ are mutually orthogonal; X_y is canonically isometric to X whereas Y_x has the metric of Y times a positive "constant" $\gamma(x)$ which varies smoothly with x.

More precisely, if $v \in T_y Y_x \subseteq T_{(x,y)}Z$, then $|v|_{(x,y)} = \gamma(x)|v|_y$, the bars denoting the obvious lengths on both sides of this equation. We also pause to remark that 'discrete' refers to the spectrum of the Friedrichs extension of $-\Delta \restriction C_0^\infty(Y)$. The Friedrichs extension provides a unified framework in which we need not distinguish between a compact manifold, say, and an open relatively compact subset with sufficiently smooth boundary of a larger manifold, both examples of discrete manifolds. In the former the Laplacian is essentially selfadjoint on C_0^∞. In the latter the Friedrichs extension corresponds to the Dirichlet problem. In connection to this it is worthwhile noting that on a complete manifold the Laplacian is always essentially selfadjoint, [4].

The manifolds under consideration will be referred to as $X \times_\gamma Y$, for short. Concerning completeness of this type of manifold we have

Proposition 3.1. *$X \times_\gamma Y$ is complete if and only if X and Y are.*

Proof. A proper submanifold of a complete Riemannian manifold is obviously complete. Since X_y and Y_x are proper submanifolds of Z and are isometric to X and Y respectively, except perhaps for a 'multiplicative constant', it is clear that they must be complete if Z is. To prove the converse it suffices to show that the closed ball of radius r about $z_0 = (x_0, y_0)$, $B_r(z_0)$, is a subset of $B_r(x_0) \times B_{r/\gamma_0}(y_0)$ where $\gamma_0 = \inf\{\gamma(x) | x \in B_r(x_0)\} > 0$. Indeed as is well known a Riemannian manifold is complete if and only if any closed ball is compact. Now the projection $Z \to X$ is distance-decreasing since for an arbitrary piecewise smooth arc $z(t) = (x(t), y(t))$, $t \in [0, 1]$, joining $z_0 = (x_0, y_0)$ to $z_1 = (x_1, y_1)$ of total length L, we have

$$\operatorname{dist}(x_0, x_1) \leqslant \int_0^1 |\dot{X}|\, dt \leqslant \int_0^1 (|\dot{x}|^2 + \gamma^2(X)|\dot{y}|^2)^{\frac{1}{2}}\, dt = L.$$

It follows that if $z_1 = (x_1, y_1) \in B_r(z_0)$, then $x_1 \in B_r(x_0)$. Now let $\varepsilon > 0$; there exists a piecewise smooth arc $z(t)$, $t \in [0, 1]$, joining z_0 to z_1 whose total length is less than $r + \varepsilon$. Consequently

$$\operatorname{dist}(x(t), x_0) \leqslant \operatorname{dist}(z(t), z_0) \leqslant \int_0^t |\dot{z}|\, dt \leqslant \int_0^1 |\dot{z}|\, dt \leqslant r + \varepsilon.$$

In other words, the projection of this arc into X lies entirely in the ball $B_{r+\varepsilon}(x_0)$, which is compact if X is complete; consequently, with $\gamma_\varepsilon = \inf\{\gamma(x) | x \in B_{r+\varepsilon}(x_0)\}$, which is positive and increases to γ_0 as ε decreases to 0, we have

$$\begin{aligned}\gamma_\varepsilon \operatorname{dist}(y_1, y_0) &\leqslant \gamma_\varepsilon \int_0^1 |\dot{y}|\, dt \leqslant \int_0^1 \gamma(x(t))|\dot{y}|\, dt \\ &\leqslant \int_0^1 (|\dot{x}|^2 + \gamma^2(x)|\dot{y}|^2)^{\frac{1}{2}}\, dt = \int_0^1 |\dot{z}|\, dt \leqslant r + \varepsilon,\end{aligned}$$

which means that $y_1 \in B_{(r+\varepsilon)/\gamma_\varepsilon}(y_0)$ from which our assertion follows by letting $\varepsilon \to 0$.

We now study the form of the Laplacian $-\Delta_Z$ as acting on $C_0^\infty(X) \otimes C_0^\infty(Y)$. This is all we need since this space is dense in $C_0^\infty(Z)$ (and therefore in $\mathfrak{D}'(Z)$). We introduce some notation as follows: $\mu = \mu(dx)$, $\nu = \nu(dy)$ the Riemannian volume elements of X, Y, respectively; m, n, the respective dimensions; $-\Delta_X$, $-\Delta_Y$, the Laplacians; $a(x, \xi)$, $b(y, \eta)$ their symbols; $g(z, \zeta)$ the symbol of $-\Delta_Z$; $0 \leqslant \lambda_1(Y) < \lambda_2(Y) \leqslant \lambda_3(Y) \leqslant \cdots$, the eigenvalues of $-\Delta_Y$ repeated according to their multiplicities. We remark that $\lambda_1(Y)$ is simple, it vanishes if Y is compact, and that the sequence $\lambda_j(Y)$ tends to ∞ as $j \to \infty$.

If $(x_1, x_2, \cdots, x_m; y_1, y_2, \cdots, y_n)$ is a coordinate system adapted to the product structure of Z, the metric tensor can be written, according to our basic assumption, in the form $ds_Z^2 = ds_X^2 + \gamma^2(x)\, ds_Y^2$, with $ds_X^2 = \Sigma\, a_{ij}(x)\, dx_i\, dx_j$ and $ds_Y^2 = \Sigma\, b_{kl}(y)\, dy_k\, dy_l$. The matrix of this tensor is therefore

$$(g_{st}) = \left[\begin{array}{c:c} (a_{ij}(x)) & 0 \\ \hdashline 0 & \gamma^2(x)(b_{kl}(y)) \end{array}\right],$$

and its determinant g satisfies

$$g = \gamma^{2n}(x)a(x)b(y), \tag{1}$$

where $a(x) = \det(a_{ij}(x))$ and $b(y) = \det(b_{kl}(y))$. Since locally the Riemannian volume is $g^{\frac{1}{2}} dz_1 \cdots dz_{n+m}$, (1) shows that (locally and therefore globally) this volume is the direct product in the sense of measure theory:

$$\gamma^n \mu \otimes \nu. \tag{2}$$

The inverse of the metric tensor can be written with standard notation

$$(g^{st}) = \left[\begin{array}{c:c} (a^{ij}(x)) & 0 \\ \hdashline 0 & \gamma^{-2}(x)(b^{kl}(y)) \end{array}\right]. \tag{3}$$

Consider the canonical decomposition $T_z^* Z = T_x^* X \oplus T_y^* Y$, $z = (x, y)$. If $(z, \zeta) \in T_z^* Z$ and $\zeta = \xi \oplus \eta$, we write $(z, \zeta) = ((x, \xi); (y, \eta))$.

(3) tells us that the symbol of the Laplacian has the form

$$g(z, \zeta) = a(x, \xi) + \gamma^{-2}(x) b(y, \eta). \tag{4}$$

Let A_0 be the operator on X (in the sense of §2) symmetric relative to $\gamma^n \mu$, whose symbol is $a(x, \xi)$ and such that $A_0(1) = 0$.

Proposition 3.2. *The Laplacian* $-\Delta_Z \restriction C_0^\infty(X) \otimes C_0^\infty(Y)$ *has the form*

$$-\Delta_Z = A_0 \otimes 1_Y + \gamma^{-2} \otimes -\Delta_Y. \tag{5}$$

Proof. A simple calculation in local coordinates gives (5). Alternatively, the operator $A = A_0 \otimes 1_Y + \gamma^{-2} \otimes -\Delta_Y$ is clearly symmetric relative to $\gamma^n\mu \otimes \nu$, its symbol is $a(x, \xi) + \gamma^{-2}(x)b(y, \eta)$ and $A(1) = 0$.

Besides A_0 we must consider operators

$$A_i = A_0 + \gamma_i(Y)\gamma^{-2}, \qquad i = 1, 2, \cdots . \tag{6}$$

The A_i are symmetric relative to $\gamma^n\mu$, they satisfy $A_i(1) = \lambda_i(Y)\gamma^{-2}$, and their symbols coincide with that of $-\Delta_X$.

For example the complement of zero in $\mathbf{R}^{1+n}$ has the form $\mathbf{R}^+ \times_r S^n$, $r(x) = |x|$, and $A_0 = r^{-n}(d/dr)r^n(d/dr)$. If $n = 1$, $A_i = -r^{-1}(d/dr)r(d/dr) + i^2r^{-2}$ is essentially the Bessel operator of order i, although for obvious reasons our numbering here of the eigenvalues of S^1 is at variance with the rest of the text.

We are now ready for the main result of this paper.

Theorem 3.3. *A necessary and sufficient condition for $Z = X \times_\gamma Y$ to be discrete is that the A_1-manifold X be discrete. If Y is compact (more generally if $\lambda_1(Y) = 0$), then Z is discrete iff the A_0-manifold X is discrete. Thus in this case, whether Z is discrete or not, only depends on X and γ, not on Y. In general, the discreteness of Z depends on Y only through $\lambda_1(Y)$.*

Proof. Let u_i be an orthonormal basis of $L^2(Y; \nu)$ made up of eigenvectors of $-\Delta_Y$ corresponding to the eigenvalues $\lambda_i(Y)$, and let D_i be the $C_0^\infty(X)$ module generated by u_i in $L^2(Z; \gamma^n\mu \otimes \nu)$. We see from (5) and (6) that A acts on D_i as the operator A_i, if the obvious identification between D_i and $C_0^\infty(X)$ is made. The family of spaces D_i and operators A_i defines as in §1 an operator $A = \bigoplus A_i$ with domain $\bigoplus D_i$ dense in $L^2(Z; \gamma^n\mu \otimes \nu)$ whose Friedrichs extension coincides with that of $-\Delta_Z \upharpoonright C_0^\infty(Z)$. To substantiate the last statement it suffices to show that the latter is an extension of the former, since selfadjoint operators are maximally symmetric. Leaving the details of all this to the reader we can now apply Corollary 1.4. The theorem will clearly follow if from the assumption that A_1 is discrete we can conclude that all the A_i are, and that $\lambda_1(A_i) \to \infty$ as $i \to \infty$. That the A_i are discrete is the content of Corollary 2.6. Now consider $\lambda_1(A_i)$. We shall show that if λ_0 is any given real scalar, then $\lambda_1(A_i) \geqslant \lambda_0$ when i is sufficiently large. From Corollary 2.4 applied to A_1 we see that there exist a compact subset K of X and a positive function $f \in C^\infty(X - K)$ such that $A_1 f(x) \geqslant \lambda_0 f(x)$ on $X - K$. Let g be a positive function in $C^\infty(X)$ which coincides with f on the complement of a compact neighborhood K_1 of K, and consider $g^{-1}(x)A_i g(x) = g^{-1}A_1 g(x) + (\lambda_i(Y) - \lambda_1(Y))/\gamma^{-2}(x)$. Since $\lambda_i(Y) \to \infty$ we can find an i_0 such that on the compact set K_1, $g^{-1}A_i g \geqslant \lambda_0$ whenever $i \geqslant i_0$. The inequality satisfied by g is clearly also true outside K_1: from Theorem 2.1 we conclude that if $i \geqslant i_0$ then $\lambda_1(A_i) \geqslant \lambda_0$.

Corollary 3.4. *If Y is discrete, $\lambda_1(Y) > 0$ and $\gamma(x) \to 0$ as $x \to \infty$, then $X \times_\gamma Y$ is discrete.*

Proof. Apply Corollary 2.4 to A_1 with $f \equiv 1$.

4. Generalizations and examples

The Decomposition Principle (Corollary 2.5) can be used to study more general manifolds than the ones treated in §3. For example a manifold M with a finite number of ends in the sense of [6, p. 80], has a compact subset K such that its complement is the disjoint union of open subsets $Z_1, Z_2, \cdots, Z_p$, say. Corollary 2.5 implies that M is discrete if and only if all of the Z_i are, since the Laplacian on $M - K$ breaks down into the (finite) direct sum of the Laplacians on each one of the components. If the Z_i admit a structure of the form $X_i \times_{\gamma_i} Y_i$, then the method of §3 can be applied, and the study of the discreteness of M reduces further to the study of the manifolds X_i with appropriate operators on them. Results in this direction are most complete when the X_i are 1-dimensional. An example of such a manifold would be the interior M of a compact manifold with boundary, $M \cup \partial M$. If the components are $Y_1, Y_2, \cdots, Y_p$, say, then the collar neighborhood theorem implies that outside some compact subset K, M decomposes in the manner described above.

As our main example we shall study surfaces of revolution in $\mathbf{R}^3$ generated by revolving a meridian $(x, 0, \gamma(x))$, $x \in \mathbf{R}$, around the x-axis say; here γ is a strictly positive function in $C^\infty(\mathbf{R}^1)$. It is easy to see that such a manifold (called hereafter Z_γ) has the structure $X \times_\gamma S^1$, where X is $\mathbf{R}^1$ with the nonstandard metric given by the arc length of the curve $(x, \gamma(y))$ in $\mathbf{R}^2$: $ds^2 = (1 + \dot{\gamma}^2(x))\, dx^2$, and S^1 has the metric of the unit circle as a subset of $\mathbf{R}^2$. This surface has two ends, and we study them separately by letting x be greater than zero and less then zero respectively.

Theorem 4.1. *A necessary and sufficient condition for the surface or revolution Z_γ to be discrete, is that the expressions*

$$I_\pm(x) = \int_0^x (1 + \dot{\gamma}^2(t))^{\frac{1}{2}} \gamma^{-1}(t)\, dt \int_x^{\pm\infty} (1 + \dot{\gamma}^2(s))^{\frac{1}{2}} \gamma(s)\, ds \tag{1}$$

tend to 0 as $x \to \pm\infty$ respectively. When $\dot{\gamma}$ is bounded the $I_\pm$ can be replaced by the simpler integrals

$$J_\pm(x) = \int_0^x \gamma^{-1}(t)\, dt \int_x^\infty \gamma(s)\, ds. \tag{1a}$$

Proof. We shall only consider the case $x > 0$, the other being similar. According to Theorem 3.3 we must prove that the operator $A_0 = -\gamma^{-1}(1 + \dot{\gamma}^2)^{\frac{1}{2}}(d/dx)(1 + \dot{\gamma}^2)^{-\frac{1}{2}}(d/dx)$ is discrete on $(0, \infty)$. To simplify A_0 we introduce the variable $y = y(x)$ defined by the equation $\dot{y} = \gamma^{-1}(1 + \dot{\gamma}^2)^{\frac{1}{2}}$, $y(0) = 0$. In this new coordinate A_0 takes the form $-r^{-2}d^2/dy^2$, with $r(y) = \gamma(x(y))$. We claim that the mapping $x \to y$ is a diffeomorphism of $[0, \infty)$ onto itself. Indeed the integral $\int_0^\infty (1 + \dot{\gamma}^2)^{\frac{1}{2}}\gamma^{-1}\, dt$ is divergent if (and only if) either $\int_0^\infty |\dot{\gamma}|\gamma^{-1}\, dt = \infty$ or else $\int_0^\infty \gamma^{-1}\, dt = \infty$. If $\int_0^\infty |\dot{\gamma}|\gamma^{-1}\, dt < \infty$, $\int_0^\infty \dot{\gamma}\gamma^{-1}\, dt$ converges and therefore $\log \gamma$ has a finite limit as $x \to \infty$, which implies that γ has a nonvanishing limit as $x \to \infty$. But then $\int_0^\infty \gamma^{-1}\, dt$ must diverge which proves our claim. We are now in a position to apply the following discreteness criterion of I. S. Kats and M. G. Krein [9], (a proof of which can be found in [5, p. 93]): the operator $-r^{-2}d^2/dy^2$ has a discrete spectrum if and only if $\lim_{y\to\infty} y \int_y^\infty r^2(s)\, ds = 0$; here y is a variable which ranges through the interval $(0, \infty)$, and 0 is a regular end-point for the operator. Reverting to the original variable x, the theorem follows.

Corollary 4.2. *If a surface of revolution is discrete, then its total volume must be finite.*

Proof. We see from formula (2) of §3, or by direct calculation, that the total volume of Z_γ is $2\pi \int_{-\infty}^\infty \gamma(s)(1 + \dot{\gamma}^2(s))^{\frac{1}{2}}\, ds$. Corollary 4.2 is clearly a consequence of the finiteness of the integrals $I_\pm(x)$.

It is interesting to note that there is a wide gap between the volume of a surface of revolution being finite and its spectrum being discrete. Indeed a surface of the form Z_γ with $\gamma = |x|^{-\alpha}$ near infinity has finite volume if and only if $\alpha > 1$. Yet *no* surface of this type is discrete, for the integrals (1a) are $\sim |x|^2/(\alpha^2 - 1)$ as $x \to \infty$. The simplest examples of (noncompact, complete) discrete surfaces of revolution are provided by generators of the form $e^{-|x|^\alpha}$ when $|x|$ is large. Even in this case not all orders of exponential decay will do. In fact such a surface is discrete if *and only if* α is strictly greater than 1! Indeed repeated integration by parts shows that as $x \to \infty$

$$\int_x^\infty e^{-t^\alpha}\, dt \sim \frac{x^{1-\alpha}e^{-x^\alpha}}{\alpha}, \quad \text{and that} \int_1^x e^{t^\alpha}\, dt \sim \frac{x^{1-\alpha}e^{x^\alpha}}{\alpha} \text{ if } \alpha < 1.$$

In any case $\int_1^x e^{t^\alpha}\, dt = O(e^{x^\alpha})$. Thus we see that the integrals $J_\pm$ of (1a) satisfy

$$J_\pm(x) \sim \begin{cases} \dfrac{x^{2(1-\alpha)}}{\alpha} & \text{if } \alpha \leqslant 1, \\ O(x^{(1-\alpha)}) & \text{if } \alpha > 1, \end{cases}$$

from which our assertion follows by applying Theorem 4.1.

To close, we take a final look at the conclusion of Corollary 4.2. For general manifolds, finiteness of the volume is by no means necessary for the spectrum to be discrete. For example in $\mathbf{R}^n$ with its standard metric, a region with a sufficiently smooth boundary will be discrete for the Dirichlet problem if and only if for every positive r there is only a finite number of disjoint balls of radius r which can be placed inside the given region (Molchanov [10]; see also [5, p. 154]). For instance the subset of $\mathbf{R}^2$ $\{(x, y) | |xy| < 1\}$ has infinite volume and a discrete spectrum. On the other hand the surfaces we have been considering are all complete. Thus a natural question arises, namely: is it necessary for a complete discrete manifold to have finite volume? As we shall see the answer is in the negative; in fact we have

Proposition 4.3. *If $n \geqslant 2$ there are functions $\varphi \in C^\infty(\mathbf{R}^n)$ such that the complete manifold $\mathbf{R}^n \times_{e^\varphi} S^1$ is discrete and has infinite volume.*

Proof. The operator A_0 on $\mathbf{R}^n$ associated to such φ is $-e^{-\varphi}$ div e^φ grad. We shall assume that φ depends only on the distance to the origin r, at least for large r. We write $\varphi = \varphi(r)$, $\dot\varphi = d\varphi/dr$. With

$$\psi = \begin{cases} r^{2-n} & \text{if } n > 2, \\ \log(1/r) & \text{if } n = 2, \end{cases}$$

we have when r is large,

$$e^{-\psi}A_0 e^{\psi} = \begin{cases} (n-2)r^{1-n}(\dot\varphi + (2+n)r^{1-n}) & \text{if } n > 2 \\ r^{-1}(\dot\varphi - r^{-1}) & \text{if } n = 2. \end{cases}$$

Let φ be any function of a real variable r whose slope is bounded below by cr^p for some scalars $c > 0$ and $p > n - 1$. These conditions imply that $\lim_{x\to\infty} e^{-\psi}A_0 e^{\psi} = +\infty$. We see using Corollary 2.4 that A_0 is discrete; by Theorem 3.3 so is $\mathbf{R}^n \times_{e^\varphi} S^1$, because the fibre is compact. The proposition follows from the observation that the volume of $\mathbf{R}^n \times_{e^\varphi} S^1$ is $2\pi \int_{\mathbf{R}^n} e^\varphi \, dx = \infty$.

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